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$o(4,2)$ Operator Replacements: Geometrical Interpretation

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The method of $o(4,2)$ operator replacements relies upon a particular realization of the $o(4,2)$ Lie algebra in terms of position and momentum operators, involving a free dimensionless parameter β. The geometrical significance of the operator replacements is given. The momentum space becomes a three-dimensional sphere of radius $[\exp(\beta)]/2$ (atomic units) embedded in a four-dimensional Euclidean space. A much simpler realization of the replaced operators is obtained.

1. INTRODUCTION

The recently introduced method of $o(4,2)$ operator replacements (de Prunelé, 1992) is aimed at solving the Schrödinger equation. This equation is replaced by another equation involving a free dimensionless parameter β. For finite β values, the replaced equation can be studied using only a discrete representation of $o(4,2)$. The solutions of the replaced equation are expected to converge to the solutions of the initial Schrödinger equation when $\beta \to +\infty$. This method appears to be particularly attractive for studying the three-body Coulomb problem since the energies appear as analytical functions of $\exp(\beta)$. An analytical continuation from the exactly solvable limit $\beta \to -\infty$ to the physical limit $\beta \to +\infty$ has been performed recently for the triplet S lowest energy of helium (Ivanov and de Prunelé, 1994). This provides a purely nonvariational approach. The present paper is mainly devoted to a better understanding of the method by emphasizing the geometrical aspects. The simplicity of the realization of the operator replacements obtained in this paper [equations (2.30)–(2.35), (2.14)–(2.17), (2.24), (2.12)] should be

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contrasted with the complexity of the initial realization [equations (1.18)–(1.23), (1.1)–(1.7)].

The method of $o(4,2)$ operator replacements can be briefly summarized as follows. The starting point is the following realization of the $o(4,2)$ algebra in terms of the position and momentum operators (see, e.g., Barut and Kleinert, 1967; Fronsdal, 1967; Nambu, 1967):

$$\mathbf{a}(\beta) \equiv \exp(-\beta)[\tfrac{1}{2}\mathbf{r}p^2 - \mathbf{p}(\mathbf{r}\cdot\mathbf{p})] - \tfrac{1}{2}\exp(\beta)\,\mathbf{r} \tag{1.1}$$

$$\mathbf{l} \equiv \mathbf{r} \times \mathbf{p} \tag{1.2}$$

$$t_2 \equiv rp_r = \mathbf{r}\cdot\mathbf{p} - i \tag{1.3}$$

$$\mathbf{b}(\beta) \equiv \mathbf{a}(\beta) + \exp(\beta)\mathbf{r} \tag{1.4}$$

$$\mathbf{g} \equiv r\mathbf{p} \tag{1.5}$$

$$t_1(\beta) \equiv r[\exp(-\beta)p^2 - \exp(\beta)]/2 \tag{1.6}$$

$$t_3(\beta) \equiv r[\exp(-\beta)p^2 + \exp(\beta)]/2 \tag{1.7}$$

Atomic units are used. It is clear that we have an $o(4,2)$ Lie algebra if the commutation relations are written as

$$i[M_{\mu\nu}, M_{\rho\sigma}] = g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\nu\sigma}M_{\mu\rho} - g_{\mu\rho}M_{\nu\sigma} \tag{1.8}$$

with the diagonal g matrix defined by

$$g_{11} = g_{22} = g_{33} = g_{44} = -g_{55} = -g_{66} = 1 \tag{1.9}$$

and the correspondence between the antisymmetric matrix $M_{\mu\nu}$ and the operators of equations (1.1)–(1.7) given by

$$M_{\mu\nu} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & l_3 & -l_2 & a_1 & b_1 & g_1 \\ & 0 & l_1 & a_2 & b_2 & g_2 \\ & & 0 & a_3 & b_3 & g_3 \\ & & & 0 & t_2 & t_1 \\ & & & & 0 & t_3 \\ & & & & & 0 \end{array} \tag{1.10}$$

with μ the index of rows and ν the index of columns. The dependence on the dimensionless parameter β corresponds to a similarity transformation. Specifically, if $u(\beta)$ is any one among the 15 generators,

$$u(\beta) = \exp(i\beta t_2)\, u(0) \exp(-i\beta t_2) \tag{1.11}$$

Given a Hamiltonian H which can be expressed in terms of the operators $\mathbf{r}$, $\mathbf{p}$, $\mathbf{l}$, $\mathbf{r}/r$, r, p^2, and p_r, one considers an alternative Hamiltonian $H(\beta)$ which is obtained from $\mathbf{H}$ by the following $o(4,2)$ operator replacements:

$$r \to 2\exp(-\beta)t_3(\beta) \tag{1.12}$$

$$\mathbf{r} \to -2\exp(-\beta)\mathbf{a}(\beta) \tag{1.13}$$

$$\mathbf{p} \to \frac{\exp(\beta)}{2}\, t_3^{-1}(\beta)\mathbf{g} \tag{1.14}$$

$$\mathbf{p}^2 \to \frac{\exp(2\beta)}{2}\, [t_3^{-1}(\beta)t_1(\beta) + 1] \tag{1.15}$$

$$\frac{\mathbf{r}}{r} \to t_3^{-1}(\beta)\mathbf{b}(\beta) \tag{1.16}$$

$$p_r \to \frac{\exp(\beta)}{2}\, t_3^{-1}(\beta)t_2 \tag{1.17}$$

It is clear that the replacements can be done in several ways. For example, the result is different if one starts from $\mathbf{p}^2$ or from $\mathbf{p}\cdot\mathbf{p}$. The physical results, corresponding to the limit where $\beta \to +\infty$, are, however, expected to be identical. In practice, the choices are determined by the criterion of simplicity. Simplicity means here that the replaced operators should be as diagonal as possible in the basis introduced in Section 2. A more symmetric form of these replacements, which leads to Hermitian operators with respect to the $1/r$ scalar product, is obtained if in the replaced Schrödinger equation $[H(\beta) - E(\beta)]|\Psi(\beta)\rangle = 0$, $|\Psi(\beta)\rangle$ is changed into $\sqrt{t_3(\beta)}\,|\Psi(\beta)\rangle$ and, accordingly, $H(\beta)$ is changed into $H'(\beta) \equiv \sqrt{t_3(\beta)}H(\beta)/\sqrt{t_3(\beta)}$. The symmetric form of the $o(4,2)$ operator replacements is thus

$$r \to 2\exp(-\beta)t_3(\beta) \equiv \tilde{r}(\beta) \tag{1.18}$$

$$\mathbf{r} \to -2\exp(-\beta)\mathbf{a}(\beta) \equiv \tilde{\mathbf{r}}(\beta) \tag{1.19}$$

$$\mathbf{p} \to \frac{\exp(\beta)}{2}\frac{1}{\sqrt{t_3(\beta)}}\,\mathbf{g}\,\frac{1}{\sqrt{t_3(\beta)}} \equiv \tilde{\mathbf{p}}(\beta) \tag{1.20}$$

$$\mathbf{p}^2 \to \frac{\exp(2\beta)}{2}\left[\frac{1}{\sqrt{t_3(\beta)}}\,t_1(\beta)\,\frac{1}{\sqrt{t_3(\beta)}} + 1\right] \equiv \widetilde{\mathbf{p}^2}(\beta) \tag{1.21}$$

$$\frac{\mathbf{r}}{r} \to \frac{1}{\sqrt{t_3(\beta)}}\,\mathbf{b}(\beta)\,\frac{1}{\sqrt{t_3(\beta)}} \equiv \left(\frac{\tilde{\mathbf{r}}}{r}\right)(\beta) \tag{1.22}$$

$$p_r \to \frac{\exp(\beta)}{2}\frac{1}{\sqrt{t_3(\beta)}}\,t_2\,\frac{1}{\sqrt{t_3(\beta)}} \equiv \widetilde{p_r}(\beta) \tag{1.23}$$

The tilde denote replaced operators. [For typographical convenience, we have written $(\tilde{\mathbf{r}}/r)(\beta)$ in equation (1.22) in place of the more correct but cumbersome notation $(\widetilde{\mathbf{r}/r})(\beta)$. This will be retained in the sequel.] The operator replacements given by equations (1.12)–(1.15) or (1.18)–(1.21) were proposed recently (de Prunelé, 1992). The replacements given by equations (1.16)–(1.17) or (1.22)–(1.23) are new. The explicit choices of the operator replacements are motivated first by the requirement that the spectra of the replaced operators "converge" in some sense to the spectra of the original operators in the limit $\beta \to +\infty$. The two spectra can, however, be different. For example, the spectrum of $\tilde{r}(\beta)$ is a pure point spectrum with eigenvalues $2\exp(-\beta)n$ (n integer greater than or equal to one), and the spectrum of r, which is continuous, is the positive real line. It is clear, however, that in the limit $\beta \to +\infty$ every point of the real line can be approached arbitrarily closely by an eigenvalue of $\tilde{r}(\beta)$. It is conjectured that, when the initial operator is the sum of a kinetic part and a potential part, the two spectra are identical. This has been proved for the case of an atomic hydrogenic Hamiltonian and verified numerically for some eigenvalues of the helium atom (de Prunelé, 1992). The second motivation is that numerous commutation relations remain unchanged for all β values. This can be seen by comparing the following two matrices, whose j, k entry is the commutator of the operators at the beginning of line j and top of column k (The calculation of all these commutators is laborious, but relies only upon the basic commutators $[r_j, p_k] = i\delta_{jk}$, $[r_j, r_k] = [p_j, p_k] = 0$):

$$
\begin{array}{c} \\ l_j \\ r_j \\ r_j/r \\ p_j \\ r \\ \mathbf{p}^2 \\ p_r \end{array}
\begin{array}{c}
\begin{array}{ccccccc} l_k & r_k & r_k/r & p_k & r & \mathbf{p}^2 & p_r \end{array} \\
\left(\begin{array}{ccccccc}
ie^{jkl}l_l & ie^{jkl}r_l & ie^{jkl}r_l/r & ie^{jkl}p_l & 0 & 0 & 0 \\
 & 0 & 0 & i\delta_{jk} & 0 & 2ip_j & ir_j/r \\
 & & 0 & * & 0 & * & 0 \\
 & & & 0 & -ir_j/r & 0 & * \\
 & & & & 0 & 2ip_r & i \\
 & & & & & 0 & * \\
 & & & & & & 0
\end{array}\right)
\end{array}
\qquad (1.24)
$$

$$
\begin{array}{c|ccccccc}
 & l_k & \tilde{r}_k(\beta) & \left(\frac{\tilde{r}_k}{r}\right)(\beta) & \tilde{p}_k(\beta) & \tilde{r}(\beta) & \tilde{\mathbf{p}}^2(\beta) & \tilde{p}_r(\beta) \\
\hline
l_i & ie^{ikl}l_l & ie^{ikl}\tilde{r}_l(\beta) & ie^{ikl}\left(\frac{\tilde{r}_l}{r}\right)(\beta) & ie^{ikl}\tilde{p}_l(\beta) & 0 & 0 & 0 \\
\tilde{r}_i(\beta) & & \left(\frac{2}{z}\right)^2 ie^{ikl}l_l & -\left(\frac{2}{z}\right)^2 i\delta_{ik}\tilde{p}_r & i\delta_{ik}\left[1-\frac{2}{z^2}\tilde{\mathbf{p}}^2(\beta)\right] & 0 & 2i\tilde{p}_i(\beta) & i\left(\frac{\tilde{r}_i}{r}\right)(\beta) \\
\left(\frac{\tilde{r}_i}{r}\right)(\beta) & & & 0 & * & i\left(\frac{2}{z}\right)^2\tilde{p}_i(\beta) & * & 0 \\
\tilde{p}_i(\beta) & & & & 0 & -i\left(\frac{\tilde{r}_i}{r}\right)(\beta) & 0 & * \\
\tilde{r}(\beta) & & & & & 0 & 2i\tilde{p}_r(\beta) & i\left[1-\frac{2}{z^2}\tilde{\mathbf{p}}^2(\beta)\right] \\
\tilde{\mathbf{p}}^2(\beta) & & & & & & 0 & * \\
\tilde{p}_r(\beta) & & & & & & & 0
\end{array}
\tag{1.25}
$$

with $z \equiv \exp(\beta)$. Those commutators that cannot be expressed as a linear combination of the operators lying at the top or left of the matrices are represented by a star. One notices in particular that the components of $(\tilde{\mathbf{r}}/r)(\beta)$ commute, which motivates the choice of the replacement (1.22) in place of the replacement

$$\frac{\mathbf{r}}{r}(\beta) \rightarrow -\frac{1}{\sqrt{t_3(\beta)}}\,\mathbf{a}(\beta)\,\frac{1}{\sqrt{t_3(\beta)}}$$

which could appear more natural in view of equations (1.18), (1.19) but whose components do not commute.

An essential point to be emphasized first is that the operator replacements break the symmetry between configuration space and momentum space. Indeed, whereas it is meaningful to speak of replaced momentum space because the operators $\tilde{p}_j(\beta)$ commute, one cannot ascribe to the operators $\tilde{r}_j(\beta)$ a configuration space in the usual sense, because the do not commute.

Comparison of matrix (1.25) with the matrix (1.24) also shows that in most cases the commutation relations given by the matrix (1.24) are exactly preserved by the matrix (1.25). One observes also a so-called group contraction (Inonu and Wigner, 1953): the $o(4)$ algebra spanned by $\mathbf{l}$, $\tilde{\mathbf{r}}(\beta)$ [matrix

(1.25)] contracts as $\beta \to +\infty$ to the algebra of the group of displacements in momentum space spanned by **l**, **r** [matrix (1.24)].

The main purpose of the present paper is to give a geometrical interpretation of the replaced operators. This is done in Section 2. The equivalent form of the operator replacements obtained in equations (2.30)–(2.35) below appears to be simpler than the original form [equations (1.18)–(1.23)] and some of the properties of the replaced operators thus appear to be transparent. The essential result is that the momentum space becomes a compact space, specifically a three-dimensional sphere of radius $[\exp(\beta)]/2$ in a four-dimensional Euclidean space. A distinction from Fock's treatment of the hydrogen atom (Fock, 1935) should be noticed. Fock used the stereographic projection of the three-dimensional Euclidean momentum space onto a hypersphere as an energy-dependent change of variables. In the present approach, the points of the hypersphere *are* the points of the replaced momentum space.

This paper has three appendices. The spectral properties of the replaced kinetic operator are determined in Appendix A. Appendix B illustrates how the operator replacements yield the correct result for the matrix elements between r and p in the limit where $\beta \to +\infty$. Finally, Appendix C is not properly concerned with the operator replacements, but rather with the geometrical interpretation of some of the building blocks of the replaced operators: it is shown that the exponential of an arbitrary element of the Lie algebra $o(4, 1)$ spanned by $\mathbf{r} = \exp(-\beta)[\mathbf{b}(\beta) - \mathbf{a}(\beta)]$, $\mathbf{l}$, t_2, $\mathbf{a}(\beta)$ acts on a dimensionless function $f(\mathbf{p})$ as the exponential of a generator of a conformal transformation would act on a function $f(\mathbf{p})$ which has the dimension of $[\text{momentum}]^{-2}$. It is of course well known that $\mathbf{r} = \mathbf{b}(0) - \mathbf{a}(0)$, $\mathbf{l}$, t_2 are related to the generators of translations, rotations, and global scale changes in momentum space. The geometric significance of $\mathbf{a}(0)$ seems to be less well known. One usually attributes to $\mathbf{a}(0)$ a geometrical significance by noting that it corresponds to the reduced Runge Lenz vector for the ground state of the hydrogen atom. The geometrical interpretation is then obtained by using a Fock change of variables (Fock, 1935), i.e., an energy-dependent stereographic projection, and $\mathbf{a}(0)$ appears to be related to the generators of rotations in the planes containing the fourth dimension. We believe that it is also useful to give a purely geometrical interpretation, as done in Appendix A, without any reference to the hydrogenic Hamiltonian.

2. THE REPLACED OPERATORS: THE MOMENTUM SPACE AS A THREE-DIMENSIONAL SPHERE EMBEDDED IN A FOUR-DIMENSIONAL SPACE

The 15 operators given by equations (1.1)–(1.7) acting in the Hilbert space spanned by the so-called Sturmian functions realize a discrete unitary irreducible representation of $O(4, 2)$ (see, e.g., Adams *et al.*, 1988). Each vector of this orthonormal basis of Sturmian functions is characterized by three integers, $1 \leq n < \infty$, $0 \leq l \leq n - 1$, $-l \leq m \leq l$, and the real number β, which characterizes the length scale. We consider an abstract separable Hilbert space spanned by the orthonormal basis $|n, l, m, \beta\rangle$. The vectors of this Hilbert space are the vectors $|\Psi(\beta)\rangle = \sum_{n,l,m} c_{n,l,m}|n, l, m, \beta\rangle$ for which $\sum_{n,l,m}|c_{n,l,m}|^2$ converges. It is clear that the action of any β-dependent generator in the β-dependent basis is in fact β independent, since, corresponding to the similarity transformation of equation (1.11), one has $|n, l, m, \beta\rangle \equiv \exp(i\beta t_2)\,|n, l, m\rangle$. Thus we shall now suppress the β dependence in *both* the generators and basis vectors. The 15 generators of $o(4, 2)$ [equations (1.1)–(1.7)] can be defined equivalently by the following equations (see, e.g., Adams *et al.*, 1988):

$$l_\pm|n, l, m\rangle = c(\pm m, l)|n, l, m \pm 1\rangle \tag{2.1}$$

$$t_\pm|n, l, m\rangle = c(l, \pm n)|n \pm 1, l, m\rangle \tag{2.2}$$

$$a_3|n, l, m\rangle = [(l - m)(l + m)]^{1/2}d_l^n|n, l - 1, m\rangle + [(l - m + 1)(l + m + 1)]^{1/2}d_{l+1}^n|n, l + 1, m\rangle \tag{2.3}$$

with

$$c(a, b) \equiv [(b + a + 1)(b - a)]^{1/2} \tag{2.4}$$

$$d_l^n \equiv [(n^2 - l^2)/(4l^2 - 1)]^{1/2} \tag{2.5}$$

$$u_\pm \equiv u_1 \pm iu_2 \tag{2.6}$$

and from the commutation relations given by equations (1.8)–(1.10). In particular, one notices that the basis vectors are eigenvectors of t_3 with eigenvalues n:

$$t_3|n, l, m\rangle = n|n, l, m\rangle \tag{2.7}$$

From now on we take the viewpoint that the above actions of the generators [equations (2.1)–(2.3)] together with the commutation relations [equations (1.8)–1.10)] *define* these generators. This viewpoint corresponds to the numerical applications performed (de Prunelé, 1992; Ivanov and de Prunelé, 1994) for the helium atom. The remaining part of this section relies only upon the definitions given by equations (2.1)–(2.6) and equations (1.8)–(1.10). Thus, in this section, we ignore completely the explicit expression of the replaced operator in terms of **r** and **p** and study the replaced problem *per se*: We start from an abstract separable Hilbert space defined by the orthonormal basis $|n, l, m\rangle$ and operators defined in this Hilbert space by their action in this basis.

The spectral properties of the operator [see equation (1.21)]

$$\widetilde{\mathbf{p}^2} = \frac{\exp(2\beta)}{2}[X_1 + 1] \tag{2.8}$$

with

$$X_1 \equiv \frac{1}{\sqrt{t_3}}\, t_1 \,\frac{1}{\sqrt{t_3}} \tag{2.9}$$

are studied in Appendix A. It is shown that X_1 is bounded, has norm unity, and that its spectrum comprises only a continuous spectrum consisting of the interval $[-1, 1]$. Therefore $\widetilde{\mathbf{p}^2}$ is bounded, has norm $\exp(2\beta)$, and has only a continuous spectrum consisting of the interval $[0, \exp(2\beta)]$. As $\widetilde{\mathbf{p}^2}$ is bounded (for finite β values), it is natural to look for a compact momentum space, a surface of a four-dimensional space.

We consider a four-dimensional Euclidean space, with orthonormal coordinates $\pi_\mu = \pi^\mu$, $\mu = 1$–4, and realize the $o(4)$ algebra by rotations in the six planes associated with this coordinate system:

$$l_j = -ie^{jkl}\pi_k \frac{\partial}{\partial \pi^l} \tag{2.10}$$

$$a_j = -i\left(\pi_j \frac{\partial}{\partial \pi^4} - \pi_4 \frac{\partial}{\partial \pi^j}\right) = -\frac{\exp(\beta)}{2}\tilde{r}_j \tag{2.11}$$

Latin indices run from 1 to 3. It is well known (see, e.g., Vilenkin and Klimyk, 1993) that a complete system of functions on the sphere $\Sigma_{\mu=1}^4 (\pi_\mu)^2 = 1$ is provided by the following orthonormal functions:

$$\varphi_{n,l,m} = (-2i)^l l! \left[n \frac{(n-l-1)!}{(n+l)!}\right]^{1/2} \times C_{n-l-1}^{l+1}(\cos(\theta_3)) \sin^l(\theta_3)\sqrt{4\pi}\, Y_l^m(\theta_2, \theta_1) \tag{2.12}$$

$$Y_l^m(\theta_2, \theta_1) = (-1)^m 2^m \frac{\Gamma(m+\frac{1}{2})}{\Gamma(1/2)}\left[\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} \times C_{l-m}^{m+1/2}(\cos(\theta_2)) \sin^m(\theta_2) \exp(im\theta_1) \tag{2.13}$$

with C_n^λ the Gegenbauer polynomials defined by $(1 - 2tx + x^2)^{-\lambda} = \Sigma_{n=0}^\infty C_n^\lambda(t)x^n$ and the spherical coordinates defined by

$$\pi_1 = \sin(\theta_3)\sin(\theta_2)\cos(\theta_1) \tag{2.14}$$

$$\pi_2 = \sin(\theta_3)\sin(\theta_2)\sin(\theta_1) \tag{2.15}$$

$$\pi_3 = \sin(\theta_3)\cos(\theta_2) \tag{2.16}$$

$$\pi_4 = \cos(\theta_3) \tag{2.17}$$

with $0 \le \theta_1 < 2\pi$, $0 \le \theta_k \le \pi$, $k = 2, 3$. The orthonormality condition reads

$$\frac{1}{2\pi^2}\int_0^{2\pi} d\theta_1 \int_0^{\pi} \sin(\theta_2)\, d\theta_2 \times \int_0^{\pi} \sin^2(\theta_3)\, d\theta_3\, \overline{\varphi_{n',l',m'}}\varphi_{n,l,m} = \delta_{nn'}\delta_{ll'}\delta_{mm'} \tag{2.18}$$

Equations (2.10), (2.11) give

$$il_\pm = i\exp(\pm i\theta_1)\left[\frac{i}{\tan(\theta_2)}\frac{\partial}{\partial\theta_1} \pm \frac{\partial}{\partial\theta_2}\right] \tag{2.19}$$

$$ia_3 = -\cos(\theta_2)\frac{\partial}{\partial\theta_3} + \frac{\sin(\theta_2)}{\tan(\theta_3)}\frac{\partial}{\partial\theta_2} \tag{2.20}$$

The choice for the phase convention in equations (2.12), (2.13) is consistent with those of equations (2.1), (2.3):

$$l_\pm\varphi_{n,l,m} = c(\pm m, l)\varphi_{n,l,m\pm 1} \tag{2.21}$$

$$a_3\varphi_{n,l,m} = [(l-m)(l+m)]^{1/2} d_l^n \varphi_{n,l-1,m} + [(l-m+1)(l+m+1)]^{1/2} d_{l+1}^n \varphi_{n,l+1,m} \tag{2.22}$$

Moreover (Vilenkin and Klimyk, 1993)

$$\cos(\theta_3)\varphi_{n,l,m} = \frac{1}{2}\left[\frac{c(l,-n)}{\sqrt{n(n-1)}}\varphi_{n-1,l,m} + \frac{c(l,n)}{\sqrt{n(n+1)}}\varphi_{n+1,l,m}\right] \tag{2.23}$$

Comparing equation (2.23) with equations (2.2), (2.7), (2.9), one deduces that X_1 [equation (2.9)] can be identified with $\cos(\theta_3)$ if the action of t_3 on the basis functions is defined according to equation (2.7) by

$$t_3\varphi_{n,l,m} = n\varphi_{n,l,m} \tag{2.24}$$

Equation (2.24) is in fact a consequence of equations (2.19), (2.20) since the definitions (1.1), (1.2), (1.7) lead to the relation $t_3^2 = \mathbf{a}\cdot\mathbf{a} + \mathbf{l}\cdot\mathbf{l} + 1$ and since the functions $\varphi_{n,l,m}$ satisfy the differential equation (Vilenkin and Klimyk, 1993) $[\mathbf{a}\cdot\mathbf{a} + \mathbf{l}\cdot\mathbf{l} + 1]\varphi_{n,l,m} = n^2\varphi_{n,l,m}$. The explicit form of this differential equation is given by equation (2.36) below. From equation (2.8) one obtains

$$\widetilde{\mathbf{p}^2}(\beta) = \frac{\exp(2\beta)}{2}[1 + \cos(\theta_3)] = \left[\exp(\beta)\cos\left(\frac{\theta_3}{2}\right)\right]^2 \tag{2.25}$$

Now, the commutation relations [see equation (1.25)] $2i\tilde{p}_j(\beta) = [\widetilde{r_j}, \widetilde{\mathbf{p}^2}]$, $1 \le j \le 3$, and equations (2.11) and (2.25) yield

$$\widetilde{p_j} = \frac{\exp(\beta)}{2}\left[\pi_j \frac{\partial}{\partial \pi^4} - \pi_4 \frac{\partial}{\partial \pi^j}, 1 + \pi_4\right] = \frac{\exp(\beta)}{2}\pi_j \tag{2.26}$$

with the π_j defined by equations (2.14)–(2.17).

From the relation

$$\exp(i\alpha t_3)\frac{1}{\sqrt{t_3}} t_1 \frac{1}{\sqrt{t_3}} \exp(-i\alpha t_3) = \frac{1}{\sqrt{t_3}}[\cos(\alpha)\, t_1 - \sin(\alpha)\, t_2]\frac{1}{\sqrt{t_3}} \tag{2.27}$$

one further deduces

$$\widetilde{p_r}(\beta) = \frac{\exp(\beta)}{2}(-i)^{t_3}\cos(\theta_3)\,(i)^{t_3} \tag{2.28}$$

Finally, from the commutation [see equation (1.25)]

$$i\widetilde{\left(\frac{r_j}{r}\right)} = [\widetilde{r_j}, \widetilde{p_r}]$$

one obtains

$$\widetilde{\left(\frac{r_j}{r}\right)}(\beta) = (-i)^{t_3}\pi_j(i)^{t_3} \tag{2.29}$$

One should not be surprised that the right-hand side of equation (2.29) does not exhibit an explicit β dependence since this dependence does not appear in t_3 as explained at the beginning of this section, and as will be further discussed for a specific example at the end of this section.

To summarize, it has been shown that the operator replacements given by equations (1.18)–(1.23) are equivalent to the following ($j = 1, 2, 3$):

$$r \to 2\exp(-\beta)\, t_3 \equiv \tilde{r}(\beta) \tag{2.30}$$

$$r_j \to 2i\exp(-\beta)\left(\pi_j \frac{\partial}{\partial \pi^4} - \pi_4 \frac{\partial}{\partial \pi^j}\right) \equiv \widetilde{r_j}(\beta) \tag{2.31}$$

$$p_j \to \frac{\exp(\beta)}{2} \pi_j \equiv \widetilde{p_j}(\beta) \tag{2.32}$$

$$\mathbf{p}^2 \to \frac{\exp(2\beta)}{2} (1 + \pi_4) \equiv \widetilde{\mathbf{p}^2}(\beta) \tag{2.33}$$

$$\frac{r_j}{r} \to (-i)^{t_3} \pi_j (i)^{t_3} \equiv \widetilde{\left(\frac{r_j}{r}\right)}(\beta) \tag{2.34}$$

$$p_r \to \frac{\exp(\beta)}{2} (-i)^{t_3} \pi_4 (i)^{t_3} \equiv \widetilde{p_r}(\beta) \tag{2.35}$$

with π_j defined by equations (2.14)–(2.17) and t_3 defined by equation (2.24).

A simple expression for t_3 in terms of the angles θ_j seems difficult to obtain. From the differential equation (Vilenkin and Klimyk 1993)

$$\left[\frac{d^2}{d\theta_3^2} + \frac{2}{\tan(\theta_3)} \frac{d}{d\theta_3} - \frac{l_j l^j}{\sin^2(\theta_3)} + n^2 - 1\right] \varphi_{n,l,m} = 0 \tag{2.36}$$

one has

$$t_3^2 = 1 - \left[\frac{d^2}{d\theta_3^2} + \frac{2}{\tan(\theta_3)} \frac{d}{d\theta_3} - \frac{l_j l^j}{\sin^2(\theta_3)}\right] \tag{2.37}$$

The geometrical interpretation of the operator replacements is now clear: The three-dimensional Euclidean momentum space is replaced by a three-dimensional sphere of a four-dimensional Euclidean space, with radius $[\exp(\beta)]/2$. The replaced coordinate operators [equation (2.31)] are the generators of rotations in the planes containing the fourth dimension. The first three Cartesian coordinates of a point of this sphere correspond to the replaced momentum coordinates. The fourth one is related to the replaced square momentum through equation (2.33). From the relations

$$\left[\exp(\beta) \cos\left(\frac{\theta_3}{2}\right)\right]^2 = \widetilde{\mathbf{p}^2}(\beta) \neq \tilde{p}_j(\beta)\tilde{p}^j(\beta) = \left[\frac{\exp(\beta)}{2} \sin(\theta_3)\right]^2 \tag{2.38}$$

one sees that the usual relation between the kinetic energy and the momentum is not satisfied. For finite momenta, this relation is, however, satisfied in the physical limit where $\beta \to +\infty$. In this limit, equation (2.33) shows indeed that θ_3 must go to π in order that the kinetic term remain finite. Thus, for $\theta_3 = \pi - \epsilon$, one has, to second order in ϵ,

$$\widetilde{\mathbf{p}^2}(\beta) = \left[\frac{\exp(\beta)}{2}\epsilon\right]^2 + \cdots = \tilde{p}_j(\beta)\tilde{p}^j(\beta)$$

More specifically, in the limit of infinite radius of the sphere, a neighborhood of finite volume of the point characterized by $\theta_3 = \pi$ becomes indistinguishable from a neighborhood of this point belonging to the plane tangent at this point to the sphere and thus the Euclidean geometry is recovered.

Several properties of the replaced operators are transparent from the realization given by equations (2.30)–(2.35). For example, the components of the replaced momentum [equation (2.32)] and of the replaced radial momentum [equation (2.35)] have a norm equal to [exp(β)]/2. The replaced square momentum operator [equation (2.33)] has a norm equal to exp(2β), as shown previously (Appendix A), and the components of the replaced coordinates position unit vector [equation (2.34)] have a norm equal to unity.

Some other properties, however, are not immediate. For example, we believe it is instructive to give the reasons that the scalar product

$$S \equiv \sum_j \widetilde{\left(\frac{r_j}{r}\right)}(\beta)\widetilde{\left(\frac{r_j}{r}\right)}(\beta) = (-i)^{l_3} \sin^2(\theta_3)\ (i)^{l_3}$$

should act as the unit operator when $\beta \to +\infty$. Let $\psi(\theta_j)$ be a function describing a particle with finite mean value of kinetic energy. Then, as explained above, this function should concentrate in a near vicinity of the direction $\theta_3 = \pi$ as $\beta \to +\infty$. It is clear that S can be considered to act as the unit operator on this function if it can be shown that the function $(i)^{l_3}\psi(\theta_j)$ is concentrated in a near vicinity of the direction $\theta_3 = \pi/2$. We briefly sketch the qualitative arguments. For the sake of definiteness, we suppose that the function $\psi(\theta_j)$ belongs to a subspace of fixed values of l, m. It is then proportional [see equation (2.12)] to the function

$$f(\theta_3) = \sum_n c_n f_n(\theta_3)$$

with

$$f_n(\theta_3) \equiv \left[n\frac{(n-l-1)!}{(n+l)!}\right]^{1/2} C_{n-l-1}^{l+1}(\cos(\theta_3))\sin^l(\theta_3)$$

For $\theta_3 = \pi - \epsilon$, $\epsilon > 0$ arbitrarily small, the f_n functions alternate in sign according to the parity of n. Thus the sign of the coefficients c_n should also alternate, as n increases by unit steps, if constructive interferences occur in the vicinity of $\theta_3 = \pi$ (for simplicity we assume that the c_n are real). Let us now consider the function

$$(i)^{t_3}f(\theta_3) = \sum_k (-1)^k c_{2k} f_{2k}(\theta_3) + i(-1)^k c_{2k+1} f_{2k+1}(\theta_3)$$

One sees that the constructive interferences of $\psi(\theta_j)$ at the point $\theta_3 = \pi$ are converted into destructive interferences for the function $(i)^{t_3}\psi(\theta_j)$ at the same point. At the point $\theta_3 = \pi/2$, the functions f_n present successively, as n increases by unit steps, a zero, a maximum, a zero (with an alternating slope), and a minimum. From these considerations one can infer that the destructive interferences of the function $\psi(\theta_j)$ at this point are converted into constructive interferences for the function $(i)^{t_3}\psi(\theta_j)$ in the vicinity of the same point. The reason that the conversion of destructive into constructive interferences should not occur in regions different from a neighborhood of $\theta_3 = \pi/2$ is that the point $\theta_3 = \pi/2$ is the only fixed point (with respect to n) for which $|f_n|$ has an extremum [except of course the points $\theta_3 = 0, \pi$; since $f_n(\theta_3)$ is of constant sign (with respect to n) near the point $\theta_3 = 0$, it is seen that $(i)^{t_3}\psi(\theta_j)$ also presents destructive interferences near this point]. The qualitative discussion above does not pretend to rigor, but illustrates that a proper comprehension of the new realization of operator replacements may require the explicit form of the basis functions [equation (2.12)].

3. CONCLUDING REMARKS

The correspondence between initial operators and replaced operators induces a correspondence between initial vectors and replaced vectors: the eigenvectors of commuting initial operators are replaced by the eigenvectors of commuting replaced operators. This is illustrated in Appendix B for the improper eigenvectors of kinetic energy in a subspace of fixed angular momentum and for the improper eigenvectors of r in the same subspace.

An important property of the $o(4, 2)$ operator replacement method is that the replaced kinetic operator can then be expressed as $\exp(2\beta)$ multiplied by a *bounded* operator. For the nonrelativistic helium Hamiltonian, the replaced kinetic term is dominated, in the limit where $\beta \to -\infty$, by the replaced potential energy term. Hence the problem becomes exactly solvable in this limit because a basis which diagonalizes these replaced potential energy terms can be determined (for the singlet S symmetry, some eigenvalues of the replaced potential energy are infinite and therefore the present method is not directly applicable) (Ivanov and de Prunelé, 1994). Then perturbation expansion with respect to the nondiagonal part of the replaced kinetic operator can be carried out. It is the above-mentioned boundness properties that ensure (Rellich, 1969; Kato, 1984) a nonzero radius of convergence for the Rayleigh Schrödinger series in power of $\exp(\beta)$, and thus allow analytic continuation from the limit $\beta \to -\infty$ to the physical limit $\beta \to +\infty$ to be performed (Ivanov and de Prunelé, 1994).

Concerning the equivalence of the spectrum of an initial Hamiltonian with the limit of the spectrum of the replaced Hamiltonian when $\beta \rightarrow +\infty$, the present situation is the following: The equivalence is proved for an initial hydrogenic Hamiltonian by solving explicitly the replaced equation for every β value (de Prunelé, 1992). The eigenvalues are

$$\frac{\exp(\beta)}{4}\,[\exp(\beta) - \sqrt{\exp(2\beta) + (2Z/n)^2}]$$

and thus tend to the hydrogenic ones, $(-1/2)(Z/n)^2$ (Z the nuclear charge) when $\beta \rightarrow +\infty$. The replacements used for that case are given by equations (1.18) and (1.21). The numerical results obtained by de Prunelé (1992) and Ivanov and de Prunelé (1994) indicate that this equivalence should be true for a two-electron atomic Hamiltonian. (The generalization of the method to the particular case of singlet S symmetry is now in progress.) The replacements used for that case are given by equations (1.18), (1.19), and (1.21). More accurate numerical results would allow one to depart from the infinite-nuclear-mass approximation and thus the replacement (1.20) could also be tested. The determination of the necessary and sufficient conditions to be satisfied by the initial Hamiltonian for the equivalence between the two spectra is an open problem.

An essential property of the method of operator replacements is that the replaced momentum space becomes compact, specifically a sphere. The expansion of an arbitrary square-integrable function on this space, which is equivalent to $SO(4)/SO(3)$ as a homogeneous space, involves therefore only discrete summation over spherical functions. Such a discrete summation is the analog of the Fourier integral in the Euclidean case. This illustrates a connection arising between compact momentum space and the existence of a fundamental length scale. In the present model, this connection is given by equation (B.5a). An interesting question is whether such a kind of connection has something to do with the description of the real world at a more fundamental level.

APPENDIX A. SPECTRAL PROPERTIES OF THE REPLACED KINETIC ENERGY OPERATOR

Equation (2.8) shows that the replaced kinetic energy operator $p^2/(2m)$ is simply related to the operator X_1 defined by equation (2.9). This appendix is devoted to the study of the spectral properties of the operator X_1:

$$X_1 \equiv \frac{1}{\sqrt{t_3}}\,t_1\,\frac{1}{\sqrt{t_3}} = \frac{1}{2}\left(\frac{1}{\sqrt{t_3}}\,t_+\,\frac{1}{\sqrt{t_3}} + \frac{1}{\sqrt{t_3}}\,t_-\,\frac{1}{\sqrt{t_3}}\right) \equiv \frac{1}{2}\,\{X_+ + X_-\} \quad \text{(A.1)}$$

$$\|X_1\| \leq \frac{1}{2}\,\{\|X_+\| + \|X_-\|\} \quad \text{(A.2)}$$

$$\|X_+\| = \sup_{\|v\|=1} \|X_+ v\| \tag{A.3}$$

As the operators $t_j, j = 1, 2, 3$, all commute with the angular momentum operator **l**, one can work in a subspace spanned by the vectors $|n, l, m\rangle$ with fixed l, m values. We then introduce the notations $k \equiv n - l - 1$, $e_k \equiv |n = k + l + 1, l, m\rangle$. Now, if $v = \Sigma_{k=0}^{\infty} v_k e_k$ with $\Sigma_{k=0}^{\infty} |v_k|^2 = 1$, one has $\|X_+ v\|^2 = \Sigma_{k=0}^{\infty} |v_k \alpha_k|^2$ with

$$\alpha_k = \frac{c(l, k + l + 1)}{[(k + l + 2)(k + l + 1)]^{1/2}} \tag{A.4}$$

and $c(a, b)$ defined by equation (2.4). The relation

$$(\alpha_k)^2 - (\alpha_{k-1})^2 = \frac{2l(l + 1)}{(k + l - 1)(k + l)(k + l + 1)} \tag{A.5}$$

shows that α_k is an increasing sequence for $l \neq 0$ and a constant sequence for $l = 0$. Moreover, $\lim_{k\to\infty} \alpha_k = 1$, and thus $\|X_+\| \leq 1$. One shows by the same method that $\|X_-\| \leq 1$. Therefore $\|X_1\| \leq 1$. To show that the norm is equal to unity, one considers the sequence of vectors x_j of norm unity, $x_j = (j + 1)^{-1/2} \Sigma_{k=j}^{2j} e_k$; we have

$$\begin{aligned}
\|X_1 x_j\|^2 &= \frac{1}{4(j+1)} \left\| \sum_{k=j}^{2j} \alpha_{k-1} e_{k-1} + \alpha_k e_{k+1} \right\|^2 \\
&= \frac{1}{4(j+1)} \left\| \alpha_{j-1} e_{j-1} + \alpha_j e_j + \alpha_{2j-1} e_{2j} + \alpha_{2j} e_{2j+1} \right. \\
&\quad \left. + \sum_{k=j+1}^{2j-1} (\alpha_k + \alpha_{k-1}) e_k \right\|^2 \\
&> \frac{1}{4(j+1)} \sum_{k=j+1}^{2j-1} |\alpha_k + \alpha_{k-1}|^2 > \frac{1}{4(j+1)} \sum_{k=j+1}^{2j-1} |2\alpha_{k-1}|^2 \\
&> \frac{1}{4(j+1)} \sum_{k=j+1}^{2j-1} 4|\alpha_j|^2 = |\alpha_j|^2 \frac{j-1}{j+1}
\end{aligned}$$

Thus $\lim_{j\to\infty} \|X_1 x_j\|^2 = 1$ and therefore the norm of X_1 is equal to unity. [One shows by the same method that the norm of $X_2 \equiv (1/\sqrt{t_3}) t_2 (1/\sqrt{t_3})$ is also equal to unity.] This allows us to conclude (see, e.g., Roman, 1975) that the spectrum lies inside a circle of radius unity. Taking into account both boundness and Hermitian properties, one deduces (see, e.g., Roman, 1975) in particular that the spectrum must belong to the real axis, and that the residual spectrum is empty. To determine further the spectrum in the interval $[-1, +1]$, it is most convenient to make the connection with orthogonal polynomials. For that we first suppose that there exists one eigenvector of

X_1, to be denoted $|x, l, m\rangle$, with eigenvalue x: $X_1|x, l, m\rangle = x|x, l, m\rangle$. Inserting the expansion

$$|x, l, m\rangle = \sum_{n=l+1}^{\infty} P^l_{n-l-1}(x)|n, l, m\rangle \tag{A.6}$$

into the eigenvalue equation, one obtains the following three-term recursion relation for the coefficients $P^l_k(x)$:

$$b_{k-1}P^l_{k-1}(x) - xP^l_k(x) + b_k P^l_{k+1}(x) = 0 \tag{A.7}$$

with

$$b_k = \alpha_k/2 \tag{A.8}$$

Equation (A.7) together with the convention $P^l_0(x) = 1$ uniquely determine the polynomials $\{P^l_k(x)\}$ since $b_{-1} = 0$.

It is known (see, e.g., Dombrowski, 1990) that if $b_k > 0$, there exists a measure μ with respect to which the polynomials $\{P_k\}$ are orthogonal. Moreover, if $\Sigma\ (1/b_k) = \infty$, the measure is unique. This measure also gives the spectral measure of the operator X_1. Finally, as the sequence b_k is monotone increasing and converges to 1/2, it can be deduced (Dombrowski, 1990) that there is no eigenvalue in the interval $[-1, 1]$ which must be in the spectrum. It remains to determine the measure. This can be done most simply by comparing the three-term recursion relation (A.7) with the known recursion relations of the Gegenbauer (also called ultraspherical) polynomials. One obtains

$$P^l_k(x) = \left[\frac{(2l+1)!\, k!\, (k+l+1)}{(l+1)(k+2l+1)!}\right]^{1/2} C^{l+1}_k(x) \tag{A.9}$$

The following orthonormalization condition holds:

$$\int_{-1}^{1} P^l_{k'}(x)P^l_k(x)f(x)\, dx = \delta_{k'k} \tag{A.10}$$

with

$$f_l(x) = \frac{1}{\pi} 2^{2l+1} \frac{(l+1)(l!)^2}{(2l+1)!} (1 - x^2)^{l+1/2} \tag{A.11}$$

To summarize, the spectrum of X_1 consists only of a continuous spectrum and is the interval $[-1, 1]$. The expression of X_1 in term of its associated spectral resolution of the identity is $X_1 = \int x\, dE(x)$ with the j, k matrix element of E given by

$$E_{j,k}(\lambda) = \int_{-1}^{\lambda} P_j^l(x)P_k^l(x)f_l(x)\,dx$$

One has both the generalized orthogonality and closure relations

$$\langle x', l, m | x, l, m\rangle = \frac{\delta(x - x')}{f_l(x)} \tag{A.12}$$

$$1 = \int_{-1}^{1} dx\, f_l(x) | x, l, m\rangle\langle x, l, m | \tag{A.13}$$

which hold inside a subspace with l, m fixed, and for $-1 \leq x \leq 1$.

APPENDIX B. CORRESPONDENCE BETWEEN MATRIX ELEMENTS

This appendix shows by explicit calculation that the quantum result

$$\langle r, l, m | p, l, m\rangle = i^l \frac{J_{l+1/2}(rp)}{\sqrt{rp}} \tag{B.1}$$

with J the regular Bessel function as defined by Abramowitz and Stegun (1965), is preserved by the method of $o(4,2)$ operator replacement in the limit where $\beta \to +\infty$. Equation (B.1) corresponds to the following choice of normalization for the improper eigenvectors of r, p in a subspace with fixed l, m:

$$1 = \int_0^{\infty} r^2\, dr\, | r, l, m\rangle\langle r, l, m | \tag{B.2a}$$

$$1 = \int_0^{\infty} p^2\, dp\, | p, l, m\rangle\langle p, l, m | \tag{B.2b}$$

$$\langle r, l, m | r', l, m\rangle = \frac{\delta(r - r')}{r^2} \tag{B.3a}$$

$$\langle p, l, m | p', l, m\rangle = \frac{\delta(p - p')}{p^2} \tag{B.3b}$$

From equations (1.21) and (A.6), one has the correspondence between the elements of the spectra:

$$p^2 = \frac{\exp(2\beta)}{2}(x + 1) \tag{B.4a}$$

Comparison of equation (B.3b) with equation (A.12) gives

$$|p, l, m\rangle \to i^l \frac{2}{\exp(\beta)} \left[\frac{f_l(x)}{p}\right]^{1/2} |x, l, m\rangle \qquad \text{(B.4b)}$$

The arbitrary phase factor i^l is introduced for future convenience. In the same way, from equations (1.18) and (2.7), one has the correspondence

$$r = 2\exp(-\beta)\, n \qquad \text{(B.5a)}$$

This equation illustrates that for finite β values, the continuous variation of r is replaced by a discrete one, namely by steps of $2\exp(-\beta)$. Comparison of equation (B.2a) with the closure relation $1 = \sum_{n=l+1}^{\infty} |n, l, m\rangle\langle n, l, m|$ gives

$$|r, l, m\rangle \to (-1)^{l+1-n} \left(\frac{\exp(\beta)}{2}\right)^{3/2} \frac{1}{n} |n, l, m\rangle \qquad \text{(B.5b)}$$

The phase factor $(-1)^{l+1-n}$ has been introduced in order to correspond to the phase convention in equation (B.1), as will be seen in equation (B.11) below. Relations (B.4) and (B.5) allow us to study the replacement of the scalar product given by equation (B.1), especially in the physical limit where $\beta \to +\infty$. As r and p are considered to be fixed, equations (B.4a) and (B.5a) can be used to eliminate β, leading to a relation between x, p, r, n:

$$x = -1 + \frac{1}{2}\left(\frac{rp}{n}\right)^2 \qquad \text{(B.6)}$$

Equation (B.5a) shows that n is determined by r, β. We shall now take n as independent variable and the limit of infinite β becomes equivalent to the limit of infinite n. From equations (B.4), (B.5), and (A.6)

$$\langle r, l, m | p, l, m\rangle \to i^l (-1)^{l+1-n} \left[\frac{f_l(x)}{rpn}\right]^{1/2} P^l_{n-l-1}\left(-1 + \frac{1}{2}\left(\frac{rp}{n}\right)^2\right) \qquad \text{(B.7)}$$

From the relations between Jacobi and Gegenbauer polynomials (see, e.g., Abramowitz and Stegun, 1965)

$$P_n^{\alpha-1/2,\alpha-1/2}(z) = \frac{\Gamma(2\alpha)\Gamma(\alpha + n + 1/2)}{\Gamma(2\alpha + n)\Gamma(\alpha + 1/2)} C_n^{\alpha}(z) \qquad \text{(B.8)}$$

and from the relations (Abramowitz and Stegun, 1965)

$$C_n^{\lambda}(z) = (-1)^n C_n^{\lambda}(-z) \qquad \text{(B.9)}$$

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} P_n^{\alpha,\beta}\left(1 - \frac{x^2}{2n^2}\right) = \left(\frac{2}{x}\right)^{\alpha} J_{\alpha}(x) \qquad \text{(B.10)}$$

one finally obtains after some calculation

$$\lim_{n\to\infty} i^l(-1)^{l+1-n}\left[\frac{f_l(-1+\frac{1}{2}(rp/n)^2)}{rpn}\right]^{1/2} P^l_{n-l-1}\left(-1+\frac{1}{2}\left(\frac{rp}{n}\right)^2\right)$$

$$= i^l \frac{J_{l+1/2}(rp)}{\sqrt{rp}} \qquad \text{(B.11)}$$

APPENDIX C. RELATION WITH CONFORMAL TRANSFORMATIONS IN THREE-DIMENSIONAL MOMENTUM SPACE

The geometrical interpretation of the ten operators **l**, t_2, **a**(0), **b**(0) spanning an $o(4,1)$ Lie algebra requires first that we recall some facts about conformal transformations [equations (C.1)–(C.23) below]. [We consider here first the case $\beta = 0$, since the case of arbitrary β is easily deduced equation (1.11).] A conformal transformation is a transformation which preserves angles. A formulation of this property can be given within the general mathematical framework of pseudo-Riemannian manifolds M, M' with metrics G, G' (see, e.g., Todorov, 1986). Briefly, the differentiable mapping ϕ from M to M' is conformal if the corresponding tangent map $d\phi$ preserves the angles. If p^i is a coordinate basis in a neighborhood of a point P of M, this means

$$G'\left(d\phi\left(\frac{\partial}{\partial p^i}\right), d\phi\left(\frac{\partial}{\partial p^j}\right)\right) = \Omega^2(P)G\left(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}\right) \qquad \text{(C.1)}$$

if $p'^i = \phi^i(P)$ denotes a coordinate basis of the image $\phi(P)$ of the point P, one has

$$d\phi\left(\frac{\partial}{\partial p^i}\right) = \frac{\partial p'^j}{\partial p^i}\frac{\partial}{\partial p'^j}$$

and therefore

$$\frac{\partial p'^k}{\partial p^i}\frac{\partial p'^l}{\partial p^j} G'\left(\frac{\partial}{\partial p'^k}, \frac{\partial}{\partial p'^l}\right) = \Omega^2(P)G\left(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}\right) \qquad \text{(C.2)}$$

The positive function $\Omega^2(P)$ is called the conformal factor. In the simplest case to be considered here, the manifold is flat and can be covered by a single orthonormal coordinate system p^i,

$$G\left(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}\right) \equiv G_{ij} \qquad \text{(C.3)}$$

The G_{ij} form a diagonal matrice of order n,

$$G_{ii} = \begin{cases} 1 & \text{if} \quad 1 \leq i \leq p \\ -1 & \text{if} \quad p + 1 \leq i \leq n \end{cases} \tag{C.4}$$

The conformal transformations in such an n-dimensional pseudo-Euclidean space M, $G(p, q)$ $(p + q = n)$ can conveniently be constructed by considering the orthogonal transformations in a $(n + 2)$-dimensional pseudo-Euclidean space $\mathcal{M}$, $\mathcal{G}(p + 1, q + 1)$ according to the following procedure (see, e.g., Murai, 1954; Kastrup, 1962; Todorov, 1986). An orthogonal coordinate system of $\mathcal{M}$ will be denoted by η^α,

$$\mathcal{G}\left(\frac{\partial}{\partial \eta^\alpha}, \frac{\partial}{\partial \eta^\beta}\right) = G_{\alpha\beta} \tag{C.5}$$

and

$$G_{(n+1)(n+1)} = 1 = -G_{(n+2)(n+2)} \tag{C.6}$$

We take the convention that Greek indices vary between 1 and $n + 2$, Latin indices between 1 and n. The equations

$$\eta^i = \kappa p^i \tag{C.7}$$

$$\eta^{n+1} = \frac{\kappa}{2}(1 - p_j p^j) \tag{C.8}$$

$$\eta^{n+2} = \frac{\kappa}{2}(1 + p_j p^j) \tag{C.9}$$

define, for arbitrary given values of κ, p^i, a point of a cone through the origin of the space $\mathcal{M}$:

$$G_{\alpha\beta}\eta^\alpha\eta^\beta = 0 \tag{C.10}$$

One can easily compute from equations (A3.7)–(A3.9) the value of the metric $\mathcal{G}$ of $\mathcal{M}$ on the image by the tangent map of the partial derivatives with respect to p^i, p^j, the other p^k and κ being fixed:

$$\frac{\partial \eta^\alpha}{\partial p^i}\frac{\partial \eta^\beta}{\partial p^j}\mathcal{G}\left(\frac{\partial}{\partial \eta^\alpha}, \frac{\partial}{\partial \eta^\beta}\right) = \kappa^2 G_{ij} \tag{C.11}$$

Thus, for arbitrary fixed value of κ, equations (C.7)–(C.9) define a conformal mapping from the space M into the cone of the space $\mathcal{M}$. An orthogonal transformation in the space $\mathcal{M}$ is now introduced:

$$\eta'^\alpha = \Lambda^\alpha_\beta \eta^\beta \tag{C.12}$$

with the orthogonality conditions

$$G_{\alpha\beta} = \Lambda^{\gamma}_{\alpha}\Lambda^{\delta}_{\beta}G_{\gamma\delta} \tag{C.13}$$

The orthogonality conditions ensure that the point P' of coordinates η'^{α} satisfies equation (C.10), i.e., also belongs to the cone passing through the origin of the space $\mathcal{M}$. Therefore equations (C.12), (C.7)–(C.9) in turn uniquely determine new values κ', p'^{i} ($i = 1$ to n) which also satisfy equation (C.11):

$$\frac{\partial\eta'^{\alpha}}{\partial p'^{i}}\frac{\partial\eta'^{\beta}}{\partial p'^{j}}\mathcal{G}\left(\frac{\partial}{\partial\eta'^{\alpha}}, \frac{\partial}{\partial\eta'^{\beta}}\right) = \kappa'^{2}G_{ij} \tag{C.14}$$

The explicit expression of p'^{i} in terms of the p^{j} is

$$\begin{aligned} p'^{i} &= \frac{\eta'^{i}}{\eta'^{n+1} + \eta'^{n+2}} = \frac{\kappa}{\kappa'}\left(\frac{\eta'^{i}}{\kappa}\right) \\ &= \frac{\kappa}{\kappa'}\left(\Lambda^{i}_{j}p^{j} + \Lambda^{i}_{n+1}\frac{1 - p^{j}p_{j}}{2} + \Lambda^{i}_{n+2}\frac{1 + p^{j}p_{j}}{2}\right) \end{aligned} \tag{C.15a}$$

with

$$\begin{aligned} \kappa/\kappa' = \{&[\Lambda^{n+1}_{j} + \Lambda^{n+2}_{j}]p^{j} + [\Lambda^{n+1}_{n+1} + \Lambda^{n+2}_{n+1}](1 - p^{j}p_{j})/2 \\ &+ [\Lambda^{n+1}_{n+2} + \Lambda^{n+2}_{n+2}](1 + p^{j}p_{j})/2\}^{-1} \end{aligned} \tag{C.15b}$$

Comparing equations (C.11) and (C.14), one deduces that the transformation described by equations (C.15a), (C.15b) is conformal. The conformal factor is given by $(\kappa/\kappa')^{2}$. From now on, only the three-dimensional Euclidean case ($n = 3$, $G_{11} = G_{22} = G_{33} = 1$) will be considered. Nevertheless, the covariant (contravariant) characters and the summation convention will be maintained for the sake of generality and convenience.

The relation between conformal transformations and the operators $\mathbf{l}$, t_2, $\mathbf{a}(0)$, $\mathbf{b}(0)$ spanning an $o(4,1)$ algebra originates from the consideration of infinitesimal conformal transformations:

$$p'^{i} = p^{i} + \epsilon K^{i}(\mathbf{p}) + O(\epsilon^{2}) \tag{C.16}$$

$$\Omega(\mathbf{p}) = 1 + \epsilon f(\mathbf{p}) + O(\epsilon^{2}) \tag{C.17}$$

Inserting the above equations into equation (C.1), one obtains the following solutions (see, e.g., Todorov, 1986):

$$K^{i} = \tau^{i} + \omega^{i}_{j}p^{j} + \delta p^{i} - 2\chi^{j}p_{j}p^{i} + \chi^{i}p_{j}p^{j} \tag{C.18}$$

with $\omega^{i}_{j} = -\omega^{j}_{i}$ (for the present case of Euclidean space, upper and lower components are equal). An arbitrary function $F(\mathbf{p})$ therefore satisfies, to first order in ϵ,

$$F(\mathbf{p}') = F(\mathbf{p}) + \epsilon q F(\mathbf{p}) \tag{C.19}$$

$$q \equiv \tau^j \partial_j - \frac{i}{2} e_i^{jk} \omega_j^i l_k + \delta d + \chi^j k_j \tag{C.20}$$

where l_k is defined by equation (1.2), e denotes the completely antisymmetric tensor with $e^{123} = 1$, and

$$\partial_j \equiv \frac{\partial}{\partial p^j} \tag{C.21}$$

$$d \equiv p^j \partial_j \tag{C.22}$$

$$k_j \equiv p^k p_k \partial_j - 2p_j d \tag{C.23}$$

The generators ∂_j, $\mathbf{l}$, d, $\mathbf{k}$ are the generators of translations, rotations, global scale change, and special conformal transformations, respectively. The corresponding infinitesimal generators in the space $\mathcal{M}$ are easily found [see equations (C.7)–(C.9)] to be

$$L_{j5} - L_{j4} \text{ for } \partial_j$$

$$L_{j5} + L_{j4} \text{ for } k_j$$

$$L_{54} \text{ for } d$$

where $L_{\mu\nu} \equiv \eta_\mu \, \partial/\partial\eta^\nu - \eta_\nu \, \partial/\partial\eta^\mu$. The relations between the operators of equations (1.1)–(1.4) and the conformal generators are therefore

$$-it_2 = d + 2 \tag{C.24}$$

$$-ia_j(0) = \tfrac{1}{2}[-\partial_j + (k_j - 4p_j)] \tag{C.25}$$

$$-ib_j(0) = \tfrac{1}{2}[\partial_j + (k_j - 4p_j)] \tag{C.26}$$

and of course $\mathbf{l}$ belongs to both groups. The occurrence of the terms 2 in equation (C.24) and $4p_j$ in equations (C.25) and (C.26) does not allow us to interpret these operators as generators of conformal transformations. It is seen, however, that if one changes d into $d + 2$ and simultaneously $\mathbf{k}$ into $\mathbf{k} - 4\mathbf{p}$ [see equation (C.23)], then d becomes $-it_2$, $\frac{1}{2}(-\partial_j + k_j)$ becomes $-ia_j(0)$, $\frac{1}{2}(\partial_j + k_j)$ becomes $-ib_j(0)$, whereas $\mathbf{l}$ remains unchanged. (It should be noticed that the commutation relations are not only invariant under the change d into $d + 2$, but under any changes d into $d + c$, where c is any number.) As d is the generator of global change of scale, this suggests that the operator

$$\begin{aligned}&\exp(-i\theta\{\tau^j[b_j(0) - a_j(0)] + \tfrac{1}{2} e_i^{jk} \omega_j^i l_k + \delta t_2 + \chi^j[b_j(0) + a_j(0)]\}) \\ &\quad = \exp[\theta(q + 2\delta - 4\chi^j p_j)]\end{aligned}$$

acts on a dimensionless function $F(\mathbf{p})$ as the conformal transformation $\exp\{\theta q\}$ would act on a function with dimension [momentum]$^{-2}$. Specifically, one expects that

$$\{\exp[\theta(q + 2\delta - 4\chi^j p_j)]F\}(\mathbf{p}) = \left(\frac{\kappa}{\kappa'}\right)^2 F(\mathbf{p}') \tag{C.27}$$

where $\mathbf{p}'$, κ/κ' are given by equations (C.15a), (C.15b) with the matrix Λ corresponding to the orthogonal transformation in $\mathcal{M}$ space associated with the operators of conformal transformation $\exp\{\theta q\}$ in Euclidean three-dimensional space. In order to emphasize the dependence with respect to the real variable θ, we shall write $\mathbf{p}(\theta)$ in place of $\mathbf{p}'$ in the following.

Equation (C.27) is the essential result of this appendix. Proof of equation (C.27) amounts to proving that

$$\exp[\theta(q + 2\delta - 4\chi^j p_j)] \exp\{-\theta q\} = (\kappa/\kappa')^2 \tag{C.28}$$

The left-hand side of this equation, to be denoted $\mathcal{L}(\theta)$, satisfies the differential equation

$$\begin{aligned}\frac{d\mathcal{L}(\theta)}{d\theta} &= \exp[\theta(q + 2\delta - 4\chi^j p_j)]\, 2(\delta - 2\chi^j p_j) \exp\{-\theta q\} \\ &= \mathcal{L}(\theta) 2[\delta - 2\chi^j p_j(\theta)]\end{aligned} \tag{C.29}$$

Now the right-hand side of equation (C.28), the conformal factor, can be expressed according to equation (C.2) as

$$\left(\frac{\kappa}{\kappa'}\right)^2 = \Omega^2(\theta) = \sum_{k=1}^{3} \left[\frac{\partial p^k(\theta)}{\partial p^j(0)}\right]^2 \tag{C.30}$$

with arbitrary j value between 1 and 3. Using the relation [see equation (C.18)]

$$\frac{\partial p^k(\theta)}{\partial\theta} = \cdot\tau^k + \omega^{jk} p_j(\theta) + \delta p^k + \chi^k p^j(\theta) p_j(\theta) - 2\chi^j p_j(\theta) p^k(\theta) \tag{C.31}$$

one deduces after a simple calculation that

$$\frac{d\Omega^2(\theta)}{d\theta} = \Omega^2(\theta) 2[\delta - 2\chi^j p_j(\theta)] \tag{C.32}$$

Thus both members of equation (C.28) satisfy the same first-order differential equation, and, since they are obviously equal for $\theta = 0$, equation (C.28) is proved.

Equation (C.27) will now be illustrated for the simple case where q [equation (C.20)] is simply equal to $k_1 - \partial_1$. One obtains

$$\{\exp[-i\theta a_1(0)]F\}(\mathbf{p})$$
$$= [D(\mathbf{p})]^{-2}F\left(\frac{p^1\cos(\theta) - \frac{1}{2}(1 - p^jp_j)\sin(\theta)}{D(\mathbf{p})}, \frac{p^2}{D(\mathbf{p})}, \frac{p^3}{D(\mathbf{p})}\right)$$
$$\equiv [D(\mathbf{p})]^{-2}F(\mathbf{p}') \quad \text{(C.33)}$$
$$D(\mathbf{p}) \equiv p^1\sin(\theta) + \tfrac{1}{2}(1 - p^jp_j)\cos(\theta) + \tfrac{1}{2}(1 + p^jp_j) \quad \text{(C.34)}$$

The case of arbitrary value of β is then easily deduced from equation (1.11) and the relation

$$\{\exp(i\beta t_2)\, F\}(\mathbf{p}) = \exp(-2\beta)\, F(\exp(-\beta)\mathbf{p}) \quad \text{(C.35)}$$

Specifically,

$$\{\exp[-i\theta a_1(\beta)]\, F\}(\mathbf{p})$$
$$= \{\exp(i\beta t_2)\exp[-i\theta a_1(0)]\exp(-i\beta t_2)\}\, F\}(\mathbf{p})$$
$$= \exp(-2\beta)\, \{\exp[-i\theta a_1(0)\exp(-i\beta t_2)]\, F\}[\exp(-\beta)\,\mathbf{p}]$$
$$= [D(\exp(-\beta)\,\mathbf{p})]^{-2}\{\exp[-\beta(2 + it_2)]\, F\}[\exp(-\beta)\,\mathbf{p}']$$

so that finally

$$\{\exp[-i\theta a_1(\beta)]\, F\}(\mathbf{p})$$
$$= [D(\exp(-\beta)\mathbf{p})]^{-2}$$
$$\times F\left(\frac{p^1\cos(\theta) - \frac{1}{2}[\exp(\beta) - \exp(-\beta)p^jp_j]\sin(\theta)}{D(\exp(-\beta)\mathbf{p})},\right.$$
$$\left.\frac{p^2}{D(\exp(-\beta)\mathbf{p})}, \frac{p^3}{D(\exp(-\beta)\mathbf{p})}\right) \quad \text{(C.36)}$$

In particular, one verifies from equation (C.36) that, in the limit $\beta \to +\infty$, the replacement given by equation (1.19) for $\mathbf{r}$ corresponds indeed to the translation operator in momentum space:

$$\lim_{\beta\to+\infty} \{\exp[i\tau 2\exp(-\beta)\, a_1(\beta)]F\}(\mathbf{p}) = F(p^1 + \tau, p^2, p^3) \quad \text{(C.37)}$$

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